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A SIMPLIFIED SOLUTION OF THE CUBIC.*

By Emory McClintock.

In the cubic $t^3 + 3ct + d = 0$, is there not some value of d for which the substitution $t = \dfrac{v+e}{v+1}$ shall cause the coefficients of v^2 and v to vanish, thereby leading to the solution of the one cubic for which d has that special value? Performing the substitution, we have $Av^3 + 3Bv^2 + 3Cv + D$, in which we find that $B - C = (1-e)(c+e)$, and that if we put $e = -c$, we have $B = C = d - c^2 + c$, so that if d have the value $c^2 - c$, $B = C = 0$, and $D = -cA$. If, therefore, in the special cubic $t^3 + 3ct + c(c-1) = 0$, we substitute $t = \dfrac{v-c}{v+1}$, we obtain at once $Av^3 - cA = 0$, so that the solution is $v = c^{\frac{1}{3}}$.

To transform any cubic, reduced to the form

$$x^3 + 3mx + mn = 0, \tag{1}$$

to the special form thus found capable of solution, we have only to put $x = ht$. If in the special cubic we substitute $t = xh^{-1}$, we derive $x^3 + 3ch^2x + c(c-1)h^3 = 0$, which compared with (1) shows that $m = ch^2$ and $n = ch - h$. If k be written for ch, we are thus enabled to lay down the following rule for the solution of any cubic reduced to its short form.

In

$$x^3 + 3mx + mn = 0, \tag{1}$$

substitute for each letter respectively

$$x = \frac{hv-k}{v+1} = h - \frac{h+k}{v+1}, \tag{2}$$

$$m = hk, \tag{3}$$

$$n = k - h, \tag{4}$$

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* This paper was read before the American Mathematical Society at its meeting of December 28, 1900.

so that on reduction there remains

$$hv^3 - k = 0, \tag{5}$$

$$v = \sqrt[3]{\frac{k}{h}} = \sqrt[3]{\left(1 + \frac{nk}{m}\right)}. \tag{6}$$

The second forms in (2) and (6) are especially adapted for computation. There are three values of v, one for each cube root of unity. The values of h and k are obtained from (3) and (4) by solving the quadratic $m = h^2 + nh$, namely,

$$h = -\tfrac{1}{2}n \pm \sqrt{(m + \tfrac{1}{4}n^2)}, \tag{7}$$

$$k = \quad \tfrac{1}{2}n \pm \sqrt{(m + \tfrac{1}{4}n^2)}. \tag{8}$$

Since h and k have each two values, h_1 and k_1, or h_2 and k_2, such that $h_1 = -k_2$ and $h_2 = -k_1$, it follows that v has two values, such that $v_1^3 = k_1 h_1^{-1}$ and $v_2^3 = k_2 h_2^{-1} = h_1 k_1^{-1} = v_1^{-3}$, so that $v_1 = v_2^{-1}$; but this causes no corresponding duplication of x in (2), since

$$x_1 = \frac{h_1 v_1 - k_1}{v_1 + 1} = \frac{-k_2 v_2^{-1} + h_2}{v_2^{-1} + 1} = \frac{h_2 v_2 - k_2}{v_2 + 1} = x_2. \tag{9}$$

If there are two equal roots, $h + k = 0$, and the formula (2) still serves, becoming $x = h$.

The statement of (1) may of course be varied by a change of sign, as for example by writing $-3mx$ for $3mx$, when n becomes $h + k$ and other corresponding changes follow.

The simplest and least ambiguous form of Cardan's solution is $x = u - mu^{-1}$, where $u = hv$. The second expression in (2) has the advantage of presenting a cube root only once. It is believed that the second expressions in (2) and (6) are novel, while the general nature of the analysis is apparently most nearly approached by the solution of Bézout, which will be found in Matthiessen's work on literal equations. The order is, however, quite different, and the simultaneous substitution for x, m, and n, in which the present solution essentially consists, may be new, as may also be the preliminary introduction of the special cubic.